Pricing financial derivatives by a minimizing method*

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Abstract

We shall study the backward stochastic differential equation

$$Y_t = \xi + \int_t^T F(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s \ , \ t \in [0, T]$$

and we will present a new approach for the existence of its solution. This type of equation appears very often in the valuation of financial derivatives in complete markets. Therefore, the identification of the solution as the unique element in a certain Banach space where a suitably chosen functional attains its minimum becomes interesting for numerical computations.

Key words and phrases: backward stochastic differential equations, convex analysis, optimization.

2000 Mathematics Subject Classification: 65C30, 78M50.

1 Introduction. Motivation

This paper analyze some properties of backward stochastic differential equations (for short BSDE) and their applications in financial markets. This type of equations have been introduced by Pardoux, Peng in [5] and since then many researchers developed new theories and found interesting applications in a vast area of domains. Actually, BSDE are useful for the theory of contingent claim valuation, especially cases with constraints and for the theory of recursive utilities.

A solution of a BSDE is a pair of adapted processes (Y, Z) satisfying

$$(1) \qquad \begin{cases} -dY_t = F(t, Y_t, Z_t)\, dt - Z_t dW_t, \ t \in [0, T] \\ Y_T = \xi, \end{cases}$$

*The work for this paper was supported by founds from the Grant IDEI ID_395/2007.

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where F is the generator and ξ is the terminal condition. The problem is to find the price at the moment t of a contingent claim (e.g. an European option) $\xi \geq 0$, which is a contract between the broker and the dealer that pays an amount ξ at time T. In a complete market it is possible to construct a portfolio which final wealth is equal to ξ. The dynamics of the value of the replicating portfolio Y are given by (1), where Z corresponds to the hedging portfolio. In a natural way, the value at time t of the hedging portfolio is associated with the price at that moment of the financial derivative. Since there exists an infinite number of replicating portfolios (and, as a consequence, the price is not well defined), the arbitrage pricing theory imposes some restrictions on the integrability of the hedging portfolios, restrictions that are related to a risk-adjusted probability measure. Using BSDEs theory, the problem is correctly formulated (*i.e.* there exist a unique price and a unique hedging portfolio) if we restrict the admissible strategies to square-integrable ones under the primitive probability.

Let consider in a complete financial market one riskless asset B (the money market instrument), which price is given by

$$dB_t = B_t r_t dt,$$

and n basic securities $S_1, ..., S_n$ with the price modeled by

$$dS_t^i = P_t^i \left[b_t^i dt + \sum\nolimits_{j=1}^{n} \sigma_t^{i,j} dW_t^j \right],\ i = \overline{1,n}.$$

Here the predictable, bounded and nonnegative stochastic processes r, b and σ are the *short term rate*, *the stock appreciation rates*, respectively *the volatility matrix*. Moreover, there exists a predictable and bounded process θ (*the risk premium*) such that $b_t - r_t 1 = \sigma_t \theta_t$, $d\mathbb{P}\times dt$ a.s. (1 denotes the vector with all components equal to one). For a small investor the measurable processes $\pi_t^1, ..., \pi_t^n$ represents the amount of his wealth V_t invested in the ith stock at time t. The pair (V, π) (consisting in the market value and in the portfolio process) is a feasible self-financing strategy, *i.e.*.

$$dV_t = r_t V_t dt + \pi_t^* \sigma_t \left[dW_t + \theta_t dt \right]; \quad \int_0^T \left| \pi_t^* \sigma_t \right|^2 dt < +\infty; \quad V_t \geq 0,\ \forall t \in [0,T],\ \mathbb{P}\text{-}a.s.$$

Considering a positive p-integrable ($p > 1$) contingent claim ξ, there exists a hedging strategy (V, π) against ξ, that is $V_T = \xi$ and the market value V is the fair price at every $t \in [0,T]$ of the contingent claim. Identifying $Y = V$, $Z = \pi^* \sigma$ and $F(Y,Z) = -rY - Z\theta$, it is clear that, in a complete market, the price of a contingent claim or, more general, of a financial derivative is given by the unique solution of a BSDE of the form (1). In this particular case, the linear form of F leads to an explicit exponential solution

$$V_t = \mathbb{E}[\tilde{V}_T^t \xi \mid \mathcal{F}_t],\ \pi_t = (\sigma_t^*)^{-1} ((\tilde{V}_t^0)^{-1} U_t + V_t \theta_t),$$

where $\tilde{V}_{t'}^t = e^{-\left[\int_t^{t'} (r_s + \frac{1}{2}|\theta_s|^2) ds + \int_t^{t'} \theta_s^* dW_s\right]}$, $0 \leq t \leq t'$. The stochastic process U is given by the martingale representation theorem

$$\tilde{V}_t^0 V_t = \mathbb{E}(\tilde{V}_T^0 \xi) + \int_0^t U_s^* dW_s, \quad \int_0^T |U_s|^2 ds < +\infty,\ \mathbb{P}\text{-}a.s.$$

Current financial markets imposed more evolved, non-linear recursive utilities and price systems modeled by BSDE.

In the same spirit of the paper of Gyöngy and Martínez [3], our aim is to present a different approach for the solutions of the Eq.(1), realizing a new connection between stochastic analysis and convex analysis. They situated in the framework of forward stochastic differential equations. Here, we will construct, on a suitably chosen Banach space, a proper, convex, lower semicontinuous functional and we will show that there exists a correspondence between the solutions of the BSDE governing the price dynamic and the minimum point of this functional.

The paper is organized as follows. Section 2 presents some preliminaries and hypothesis that will be used during this article, while Section 3 is dedicated to the proof of the main results.

2 Preliminaries. Notations. Hypothesis

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t, W_t, t \geq 0)$ be a complete Wiener space in $\mathbb{R}^k$, *i.e.* $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, W_t, $t \geq 0$ is a standard Wiener process in $\mathbb{R}^k$ and $\mathcal{F}_t = \sigma(W_s, s \leq t) \vee \mathcal{N}$ is the natural filtration associated to the Wiener process W (it is a right continuous family of complete sub σ-algebras of $\mathcal{F}$). Here $\mathcal{N}$ represents the family of $\mathbb{P}$-null sets.

We denote by $L^r_{ad}(\Omega; C([0,T]; \mathbb{R}^d))$, $r \geq 1$ the closed linear subspace of stochastic processes $f \in L^r(\Omega, \mathcal{F}, \mathbb{P}; C([0,T]; \mathbb{R}^d))$ which are adapted to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$, *i.e.* $f(\cdot, t)$ is $\mathcal{F}_t$-measurable, $\forall\, t \in [0,T]$. The norm on this Banach space is defined by

$$\|f\|_{L^r_{ad}} = \|f\|_{r,C} \overset{def}{=} \left(\mathbb{E}\left(\sup_{t\in[0,T]} |f_t|^r\right)\right)^{1/r}$$

We also use the notation $L^r_{ad}\left(\Omega \times [0,T]; \mathbb{R}^d\right)$ to denote the Banach space of stochastic processes $f : \Omega \times [0,T] \longrightarrow \mathbb{R}^d$ such that

$$\|f\|_r \overset{def}{=} \mathbb{E}\left(\int_0^T |f_t|^r\, dt\right)^{1/r} < +\infty$$

and $f(\cdot, t)$ is $\mathcal{F}_t$-measurable a.e. $t \in [0,T]$. Observe that $L^r_{ad}\left(\Omega \times [0,T]; \mathbb{R}^d\right)$ is a closed linear subspace of $L^r\left(\Omega \times [0,T]; \mathbb{R}^d\right)$.

Consider the eq.(1), where we assume

(H_ξ) a terminal value $\xi \in L^2\left(\Omega, \mathcal{F}_T, \mathbb{P}, \mathbb{R}^d\right)$;

(H_F) a function $F : \Omega \times [0,T] \times \mathbb{R}^d \times \mathbb{R}^{d\times k} \longrightarrow \mathbb{R}^d$ satisfying, for $\eta \in L^2(\Omega \times [0,T])$ and some constants $M \in \mathbb{R}$ and $L, \gamma > 0$,

1. for every $(y, z) \in \mathbb{R}^d \times \mathbb{R}^{d\times k}$, $F(\cdot, \cdot, y, z)$ is progressively measurable and $y \longrightarrow F(\omega, t, y, z)$ is continuous;

2. $\langle F(t,y,z) - F(t,y',z), y - y'\rangle \leq M|y - y'|$;
3. $|F(t,y,z) - F(t,y,z')| \leq L|z - z'|$;
4. $|F(t,y,0)| \leq \eta_t + \gamma|t|$, $\forall\ t, y, y', z, z'$, $\mathbb{P}$-a.s. $\omega \in \Omega$.

Definition 1 *A solution of the Eq.(1) is a couple*

$$(Y,Z) \in L^2_{ad}(\Omega; C([0,T]; \mathbb{R}^d)) \times L^2_{ad}\left(\Omega \times [0,T]; \mathbb{R}^{d\times k}\right)$$

satisfying $\mathbb{P}$*-a.s.,*

$$Y_t = \xi + \int_t^T F(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \forall\ t \in [0,T].$$

It is known (see for exemple Pardoux, Peng [5] or Pardoux, Răşcanu [7]) that, under the hypothesis (H_ξ) and (H_F), there exists a unique solution of the Eq.(1). The goal of the current paper is to provide a new perspective for the proof of the existence of a solution by minimizing a suitably chosen convex functional.

Let consider the space $\mathcal{B} = L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^d) \times L^2_{ad}(\Omega \times [0,T]; \mathbb{R}^d)$ with its norm given by

$$|(\eta, f)|^2_{\mathcal{B}} \overset{def}{=} \mathbb{E}\left(|\eta|^2 + \int_0^T |f_t|^2\, dt\right)$$

and denote by $\mathcal{B}^R$ $(R > 0)$ the closed, origin-centered ball of radius R in $\mathcal{B}$. This space is a closed linear subspace of $\Lambda = L^2\left(\Omega; \mathbb{R}^d\right) \times L^2\left(\Omega \times [0,T]; \mathbb{R}^d\right)$.

Let $(y,z) = (C(\eta,f), D(\eta,f))$ be the solution of the $BSDE(\eta,f)$

$$y_t = \eta + \int_t^T f_s ds - \int_t^T z_s dW_s\ ,\ \forall\ t \in [0,T]. \tag{2}$$

that is

$$\begin{cases} y_t = \mathbb{E}\left(\eta + \displaystyle\int_t^T f_s ds \mid \mathcal{F}_t\right) \\ \eta + \displaystyle\int_0^T f_s ds = \mathbb{E}\left(\eta + \displaystyle\int_0^T f_s ds\right) - \displaystyle\int_0^T z_s dW_s. \end{cases}$$

Observe that y is a linear function on (η, f).

Define the functional $\mathcal{E} : \mathcal{B} \longrightarrow (-\infty, +\infty]$ by

$$\mathcal{E}(\eta, f) = \sup_{(\alpha,g)\in\mathcal{B}} E_{(\alpha,g)}(\eta, f), \tag{3}$$

where

$$E_{(\alpha,g)}(\eta,f) = \mathbb{E}|\eta - \xi|^2 + 2\,\mathbb{E}\int_0^T \langle y_t - u_t, f_t - F(u_t, v_t)\rangle\, dt - \mathbb{E}\int_0^T |z_t - v_t|^2\, dt,$$

with $(y,z) = (C(\eta,f), D(\eta,f))$ and $(u,v) = (C(\alpha,g), D(\alpha,g))$.

Remark 2 *If we consider, in particular,* $g_s = f_s$ *and* $\alpha = \eta$, $\forall\ s \in [0,T]$, $\mathbb{P}$*-a.s.* $\omega \in \Omega$, *we have* $E_{(\eta,f)}(\eta, f) = \mathbb{E}\left|\eta - \xi\right|^2 \geq 0$, *and, therefore,* $\mathcal{E}(\eta, f) \geq 0$, $\forall\ (\eta, f) \in \mathcal{B}$.

Some properties of the function defined above are presented in the next result.

Proposition 3 *The functional* $E_{(\alpha,g)}(\eta, f)$ *is convex and continuous as a function of* $(\eta, f) \in \mathcal{B}$ *for each fixed* (α, g). *Moreover,* $E_{(\alpha,g)}(\eta, f)$ *is continuous in* (α, g) *for any* $(\eta, f) \in \mathcal{B}$.

Proof. Applying the energy equality for the process $y - u$ we find

$$\mathbb{E}\left|y_0 - u_0\right|^2 + \mathbb{E}\int_0^T \left|z_s - v_s\right|^2 ds = \mathbb{E}\left|\eta - \alpha\right|^2 + 2\ \mathbb{E}\int_0^T \langle y_t - u_t, f_t - g_t\rangle\, dt$$

and the definition of $E_{(\alpha,g)}(\eta, f)$ yields

$$\begin{aligned} E_{(\alpha,g)}(\eta, f) &= \mathbb{E}\left|\eta - \xi\right|^2 + \mathbb{E}\left|y_0 - u_0\right|^2 - \mathbb{E}\left|\eta - \alpha\right|^2 + 2\ \mathbb{E}\int_0^T \langle y_t - u_t, g_t\rangle \\ &\quad - 2\ \mathbb{E}\int_0^T \langle y_t - u_t, F(u_t, v_t)\rangle\, dt \\ &= \mathbb{E}\left|\alpha - \xi\right|^2 + \mathbb{E}\left|y_0 - u_0\right|^2 + 2\ \mathbb{E}\langle \alpha - \xi, \eta - \alpha\rangle \\ &\quad + 2\ \mathbb{E}\int_0^T \langle y_t - u_t, g_t - F(u_t, v_t)\rangle\, dt \end{aligned}$$

Since the mapping $(\eta, f) \longrightarrow \mathbb{E}\left|y_0 - u_0\right|^2$ is convex and all the others terms depend in a linear manner on (η, f) (or (y, z)) we can conclude that the convexity take place. It follows easily that the application $(\eta, f) \longrightarrow \mathcal{E}(\eta, f)$ is also convex. The same calculus yields the continuity result and it implies that the functional $\mathcal{E}$ is also continuous (and, therefore, lower semicontinuous). ■

Next section will characterize the solution of Eq.(1) in terms of the minimizing point of the functional $\mathcal{E}$.

3 Main results

Theorem 4 *The following assertions hold:*

(*a*) *Under the hypothesis* (H_ξ) *and* (H_F), *if* $(Y_t, Z_t)_{t\in[0,T]}$ *is the solution of the backward stochastic differential equation (1), then*

$$\min_{(\eta,f)\in\mathcal{B}} \mathcal{E}(\eta, f) = \mathcal{E}(\xi, F(t, Y_t, Z_t)) = 0.$$

(*b*) *If we have that* $\min\limits_{(\eta,f)\in\mathcal{B}} \mathcal{E}(\eta, f) = 0$ *then the Eq.(1) has a solution* $(Y_t, Z_t)_{t\in[0,T]}$ *and the minimum is attained in* $(\xi, F(t, Y_t, Z_t))$.

Proof. (a) Consider $(Y_t, Z_t)_{t\in[0,T]}$ the solution of the Eq.(1). It is easy to see that the pair $(\xi, F(t, Y_t, Z_t)) \in \mathcal{B}$. Indeed, by the Lipschitz and the growth condition on F

$$\mathbb{E}\int_0^T |F(t, Y_t, Z_t)|^2 \, dt \leq 2L^2 \, \mathbb{E}\int_0^T |Z_t|^2 \, dt + \tilde{C}T\mathbb{E}\left(\sup_{t\in[0,T]} |Y_t|^2\right) < +\infty.$$

To conclude the first part it is sufficient to prove that

$$E_{(\alpha,g)}(Y_T, F(t, Y_t, Z_t)) \leq 0, \ \forall \ (\alpha, g) \tag{4}$$

We have

$$\begin{aligned} E_{(\alpha,g)}(Y_T, F(t, Y_t, Z_t)) &= 2 \, \mathbb{E}\int_0^T \langle Y_t - u_t, F(t, Y_t, Z_t) - F(t, u_t, v_t)\rangle \, dt - \mathbb{E}\int_0^T |Z_t - z_t|^2 \, dt \\ &\leq \mathbb{E}\int_0^T \left(2M|Y_t - u_t|^2 + 2L|Y_t - u_t||Z_t - v_t| - |Z_t - v_t|^2\right) dt \\ &\leq \mathbb{E}\int_0^T \left(2M + L^2\right)|Y_t - u_t|^2 \, dt \leq 0 \qquad \Longleftrightarrow \qquad M \leq -\frac{1}{2}L^2. \end{aligned}$$

We can assume, without loosing the generality that the last inequality holds. Indeed, if we denote $(\tilde{Y}_t, \tilde{Z}_t) = (e^{-\alpha t}Y_t, e^{-\alpha t}Z_t)$ (α arbitrarily) it follows that $(Y_t, Z_t) = (C_t(\xi, F), D_t(\xi, F))$, $\forall t \in [0, T]$ iff $(\tilde{Y}_t, \tilde{Z}_t) = (C_t(\tilde{\xi}, \tilde{F}), D_t(\tilde{\xi}, \tilde{F}))$ $\forall t \in [0, T]$, where

$$\begin{cases} \tilde{\xi} = e^{-\alpha T}\xi \\ \tilde{F}(t, \tilde{Y}_t, \tilde{Z}_t) = \alpha\tilde{Y}_t + e^{-\alpha t}F(t, e^{\alpha t}\tilde{Y}_t, e^{\alpha t}\tilde{Z}_t) \end{cases}$$

For the transformed equation, the condition (H_F-(2)) becomes

$$\left\langle \tilde{F}(t, \tilde{y}, \tilde{z}) - \tilde{F}(t, \tilde{y}', \tilde{z}), \tilde{y} - \tilde{y}' \right\rangle \leq (M + \alpha)|\tilde{y} - \tilde{y}'|^2$$

Considering the monotonicity constant $\tilde{M} = M + \frac{1}{2}L^2 \leq 0$, it yields (4).

(b) Let $(\eta^0, f^0) \in \mathcal{B}$ be the minimizing point of the functional $\mathcal{E}$ and (y^0, z^0) the associated process

$$y_t^0 = \eta^0 + \int_t^T f_s^0 ds - \int_t^T z_s^0 dW_s \ , \ t \in [0, T].$$

Since $\mathcal{E}(\eta^0, f^0) = 0$ it yields that $E_{(\alpha,g)}(\eta^0, f^0) \leq 0$ for every $(\alpha, g) \in \mathcal{B}$ such that $(u, v) = (C(\alpha, g), D(\alpha, g))$. Considering $(\alpha, g) = (\eta^0, f^0)$, from the formula of $E_{(\alpha,g)}(\eta^0, f^0)$ we obtain $\mathbb{E}|\eta^0 - \xi|^2 \leq 0$. Hence $\eta^0 = \xi$ a.s.

Define, for every $\lambda \geq 0$ the pair $(\hat{u}, \hat{v})$ by

$$\hat{u}_t = y_t^0 - \lambda u_t \quad \text{and} \quad \hat{v}_t = z_t^0 - \lambda v_t \ , \tag{5}$$

which can play the role of (u,v) from the formula of E. We obtain

$$2\,\mathbb{E}\int_0^T \left\langle y_t^0-\hat{u}_t, f_t^0-F\left(t,\hat{u}_t,\hat{v}_t\right)\right\rangle dt-\mathbb{E}\int_0^T \left|z_t^0-\hat{v}_t\right|^2 dt\le 0$$

Using (5) and dividing by λ

$$2\,\mathbb{E}\int_0^T \left\langle u_t, f_t^0-F\left(t,y_t^0-\lambda u_t,z_t^0-\lambda v_t\right)\right\rangle dt-\lambda\int_0^T \left|v_t\right|^2 dt\le 0,$$

inequality that is valid for every $\lambda\ge 0$. Therefore

$$\mathbb{E}\int_0^T \left\langle u_t, f_t^0-F\left(t,y_t^0-\lambda u_t,z_t^0-\lambda v_t\right)\right\rangle dt\le 0.$$

Making use of the hypothesis (H_F- $(3)\,,(4)$), by Lebesgue dominated convergence theorem, we can pass to the limit as $\lambda\to 0$ inside the integral and it yields

$$\mathbb{E}\int_0^T \left\langle u_t, f_t^0-F\left(t,y_t^0,z_t^0\right)\right\rangle dt\le 0.$$

Considering now $-u$ in the place of u, by a density result in $L^2\left(\Omega\times[0,T];\mathbb{R}^d\right)$, it follows that

$$f_t^0=F\left(t,y_t^0,z_t^0\right),\ d\mathbb{P}\times dt\ a.e.\ (\omega,t)\in\Omega\times[0,T].$$

This assertion completes the proof of Theorem 4. ■

Proposition 5 *The function $\mathcal{E}:\mathcal{B}\longrightarrow(-\infty,+\infty]$ defined by (3) always attains its minimum in $\mathcal{B}^R$, for every $R>0$.*

Proof. A well known result of convex analysis (see, e.g., Zeidler [8]) establish sufficient conditions for a functional defined on a subset $\mathcal{D}$ of a reflexiv Banach space to attain its minimum. More precisely, if the subset is convex, bounded and closed in the strong topology and the functional is weak sequentially lower semicontinuous the minimum is reached. In our particular case

- $\mathcal{E}:\mathcal{B}^R\subset\mathcal{B}\subset\Lambda\longrightarrow(-\infty,+\infty]$;
- $\mathcal{D}=\mathcal{B}^R$ is bounded, convex and closed in the strong topology;
- Λ is a reflexive Banach space.

Therefore, since $\mathcal{E}$ is a supremum of convex and continuous functions, it yields that $\mathcal{E}$ is sequentially lower semicontinuous in $\mathcal{B}^R$, which completes our proof. ■

We end this section with a result which specifies that the minimum of the functional defined by (3) is always reached. For this we will make use of the following existence result in the case when the generator function F is Lipschitz in both the coefficients y and z.

Theorem 6 *If the hypothesis* (H_ξ) *and* (H_F) *(but with* $(H_F$*-(2))* *replaced by a Lipschitz condition with respect to* y*) are satisfied, then there exists a unique solution of the* $BSDE(\xi, F)$*. Moreover,*

$$\mathbb{E}\left(\sup_{t\in[0,T]}|Y_t|^2\right)+\mathbb{E}\left(\int_0^T|Z_t|^2\,dt\right)\le C_1\;\mathbb{E}\left(|\xi|^2+\int_0^T|F\left(s,0,0\right)|^2\,ds\right),$$

where $C_1 = C_1(L)$ *is a positive constant.*

The proof is based on a fixed point theorem (see, e.g., Pardoux, Peng [5] or Pardoux, Răşcanu [7]). Since the intention is to provide the presented link between stochastic analysis and convex analysis, we will skip the proof of the above result.

Theorem 7 *Under the assumptions* (H_ξ) *and* (H_F)*, there exists a sequence* $\{(\eta^\varepsilon, f^\varepsilon)\}_{\varepsilon\in I}$ *such that*

$$lim_{\varepsilon\to 0}\mathcal{E}(\eta^\varepsilon, f^\varepsilon)=\min_{(\eta,f)}\mathcal{E}(\eta,f)=0.$$

Proof. For every $K > 0$, let define the auxiliary functions $\mathcal{E}^K : \mathcal{B} \longrightarrow(-\infty, +\infty]$ by

$$\mathcal{E}^K\left(\eta,f\right)\overset{def}{=}\sup_{(\alpha,g)\in\mathcal{B}^K}E_{(\alpha,g)}\left(\eta,f\right) \tag{6}$$

We will construct a sequence $\{(\eta^\varepsilon, f^\varepsilon)\}_{\varepsilon\in I}$ such that, for every K big enough, $\mathcal{E}^K(\eta^\varepsilon, f^\varepsilon)\to 0$ as $\varepsilon\to 0$. Then, there exists a limit point $(\hat\eta, \hat f)$ for which

$$\mathcal{E}^K(\hat\eta,\hat f)\le\liminf_{\varepsilon\to 0}\;\mathcal{E}^K(\eta^\varepsilon,f^\varepsilon)=0$$

But the functionals defined by (6) are increasing with respect to R and we obtain

$$\mathcal{E}(\hat\eta,\hat f)=\sup_{K>0}\mathcal{E}^K(\hat\eta,\hat f)\le 0.$$

Using a similar technique with the one we can find in Pardoux, Răşcanu [7], consider $(\eta^\varepsilon, f^\varepsilon) = (\xi, -F_\varepsilon(\cdot, y^\varepsilon, z^\varepsilon))$, with

$$(y^\varepsilon,z^\varepsilon)=(C(\xi,-F_\varepsilon(\cdot,y^\varepsilon,z^\varepsilon)),D(\xi,-F_\varepsilon(\cdot,y^\varepsilon,z^\varepsilon))).$$

Here

$$F_\varepsilon(t,y,z)\overset{def}{=}\frac{1}{\varepsilon}(y-J_\varepsilon(t,y,z))=-F(t,J_\varepsilon(t,y,z),z),$$

where, for every $\varepsilon\in(0,1\wedge 1/2\gamma)$, $J_\varepsilon:\Omega\times[0,T]\times\mathbb{R}^d\times\mathbb{R}^{d\times k}\to\mathbb{R}^d$ is the progressively measurable operator defined by $J_\varepsilon-\varepsilon F(t,J_\varepsilon,z)=y$.

By standard calculus, we have

$$E_{(\alpha,g)}(\xi,-F_\varepsilon(\cdot,y^\varepsilon,z^\varepsilon))\le\Theta\left(\varepsilon\right),\;\text{ where }lim_{\varepsilon\to 0}\Theta\left(\varepsilon\right)=0. \tag{7}$$

Indeed,

(8) $E_{(\alpha,g)}(\xi,-F_\varepsilon(\cdot,y^\varepsilon,z^\varepsilon)) =$

$$\begin{aligned}
&2\,\mathbb{E}\int_0^T \langle y_t^\varepsilon - u_t, -F_\varepsilon(t,y_t^\varepsilon,z_t^\varepsilon) - F\left(t,y_t^\varepsilon,z_t^\varepsilon\right) + F\left(t,y_t^\varepsilon,z_t^\varepsilon\right) \\
&-F\left(t,u_t,z_t^\varepsilon\right) + F\left(t,u_t,z_t^\varepsilon\right) - F\left(t,u_t,v_t\right)\rangle\, dt - \mathbb{E}\int_0^T \left|z_t^\varepsilon - v_t\right|^2 dt \\
&\le 2\,\mathbb{E}\int_0^T \langle y_t^\varepsilon - u_t, -F_\varepsilon(t,y_t^\varepsilon,z_t^\varepsilon) - F\left(t,y_t^\varepsilon,z_t^\varepsilon\right)\rangle\, dt \\
&+\left(2M+L^2\right)\ \mathbb{E}\int_0^T \left|y_t^\varepsilon - u_t\right|^2 dt \\
&\le 2\,\mathbb{E}\int_0^T \langle y_t^\varepsilon - u_t, -F_\varepsilon(t,y_t^\varepsilon,z_t^\varepsilon) - F\left(t,y_t^\varepsilon,z_t^\varepsilon\right)\rangle\, dt.
\end{aligned}$$

The boundedness of $y^\varepsilon, z^\varepsilon$ in $L^2_{ad}\left(\Omega; C\left([0,T];\mathbb{R}^d\right)\right)$, respectively $L^2_{ad}\left(\Omega\times[0,T];\mathbb{R}^{d\times k}\right)$ implies the convergence on a subsequence (as $\varepsilon\to 0$) to the stochastic processes y, respectively z. We also have (on a subsequence ε_n) the week convergence of $-F_{\varepsilon_n}\left(\cdot,y^{\varepsilon_n},z^{\varepsilon_n}\right)$ in $L^2\left(\Omega\times[0,T];\mathbb{R}^d\right)$. Using in (8) the energy equality corresponding to the pair $(y^\varepsilon,z^\varepsilon)$ we obtain (7).

Since f^ε is bounded in $L^2(\Omega\times[0,T];\mathbb{R}^d)$ by a constant $C_2 = C_2(L,\gamma,\eta,\xi,T)$ we consider $R_0 \overset{def}{=} \mathbb{E}\left|\xi\right|^2 + C_2$, which yields

$$\mathbb{E}\left(\sup_{t\in[0,T]}\left|y_t^\varepsilon\right|^2\right) + \mathbb{E}\left(\int_0^T \left|z_t^\varepsilon\right|^2 dt\right) \le K_0 \overset{def}{=} C_1 R_0.$$

Let $R \ge R_0$ and $K > K_0$. It is obvious that for every $\varepsilon \in (0, 1\wedge 1/2\gamma)$ and for every $(\eta,f)\in\mathcal{B}^R$ we have that $|(\eta,f)|_{\mathcal{B}} \le K$, where $(y,z) = (C\left(\eta,f\right), D\left(\eta,f\right))$. Moreover,

$$\mathcal{E}^K\left(\eta,f\right) \le 0.$$

The proof is now complete. ■

Remark 8 *We have the uniqueness of the solution for the Eq.(1) in a classical way, by applying the energy equality to $Y^1 - Y^2$, where Y^1 and Y^2 are two supposed solutions. The interested reader can find more details in various papers ([1], [4], [5]...).*

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